Original Article

Nonlinear Differential and Integro-Differential Equations with Proportional Delays by Differential Transform Method with Adomian Polynomials

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Abstract

In this paper, the differential transformation is applied to approximate and exact solutions of nonlinear integro-differential and differential equations with proportional delays. In this technique, the nonlinear term is replaced by its Adomian polynomials for k index, so the dependent variable components in the recurrence relation are replaced by their corresponding differential transform components of the same index. Therefore, the nonlinear integro-differential equation can be easily solved with less computational works for any analytical nonlinearity due to the available algorithms and properties of the Adomian polynomials. In illustrative examples, the present method is applied to a few types of nonlinearity are treated and the proposed technique has provided good results.

Keywords: Differential transformation; Nonlinear integro-differential; Adomian polynomials; Proportional delays

Introduction

Integro-differential and integral equations have found applications in engineering, biomedical engineering, physics, chemistry and biological [1-3]. Functional-differential equations with proportional delays have particularly described some models including polymer crystallization and motion of particles in liquid that can be found in [4]. There are many ways to approach for solutions of integral and integro-differential equations. For example, the linear and non-linear integro-differential equations have been solved by the Legendre wavelets method [5], the Haar functions method [6, 7], the Adomian decomposition method [8, 9], the Taylor polynomial method [10-12] and the linearization method [13]. In this paper, we consider the following integro-differential and nonlinear differential equations with proportional delays:

$$\mathcal{F}\left(t, y(p_0 t), y'(p_1 t), \cdots, y^{(n)}(p_n t)\right) = 0 \quad t \geq 0, \qquad (1)$$

$$G(t, y(p_0 t), y'(p_1 t), \cdots, y^{(n)}(p_n t), \int_0^{rt} K(t, s, y(q_0 s), y'(q_1 s), \cdots, y^{(m)}(q_m s) ds) = 0, t \geq 0 \qquad (2)$$

where $\mathcal{F}$, G, K are given functions with appropriate domains of definition, p_i, q_j, $r \in (0,1)$, $i = 0,1,2 \cdots, n$, $j = 0,1, \cdots, m$, $m < n$. We introduce a more efficient and comprehensive way to use the differential transform method (DTM) for solving integro-differential and nonlinear differential equations; the idea is based on the methodology in [14]. The nonlinear function is replaced by its Adomian polynomials and then the dependent variables are replaced by their corresponding differential transform component of the same index. This technique benefited from the properties of the efficient algorithm and Adomian polynomials to quickly generate them as in the work [15-17].

Notations and Preliminaries

A review of the differential transformation method is presented here.

Differential transformation method

In the study of electric circuits, Zhou introduced the differential transformation method in 1987 [18]. The methods

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based on yields of differential transformation and Taylor series are difference equations that solutions give the exact values of derivatives of origin function at a given point.

The main advantage of differential transformation from Fourier and Laplace transformations is that it can be easily applied to linear equations with some nonlinear equations and variable and constant coefficients.

The differential transformation of the kth derivative of function $y(t)$ is defined as follows:

$$Y(k)=\frac{1}{k!}\left[\frac{d^k y(t)}{dt^k}\right]_{t=t_0}, \quad (3)$$

where $Y(k)$ is the transformed function and $y(t)$ is the original function. Differential inverse transformation of $Y(k)$ is defined as follows:

$$y(t)=\sum_{k=0}^{\infty} Y(k)(t-t_0)^k. \quad (4)$$

Inverse transformation (4.2) implies that the concept of differential transformation is derived from the Taylor series expansion. Although DTM is not able to symbolically assess the derivatives, relative derivatives can be calculated in an iterative manner that is described by the transformed equations of the original function.

From definitions (3), (4) we can derive the following.

Theorem 2.1. Assume that $\Psi(k)$, $G(k)$, $H(k)$ and $Y_i(k)$, $i=1,\cdots,n$, are the differential transformations of the functions $\psi(t)$, $g(t)$, $h(t)$ and $y_i(t)$, $i=1,\cdots,n$, respectively, then: If $\psi(t)=\frac{d^n g(t)}{dt^n}$, then $\Psi(k)=\frac{(k+n)!}{k!}G(k+n)$.

If $\psi(t)=g(t)h(t)$, then $\Psi(k)=\sum_{l=0}^{k} G(l)H(k-l)$.

If $\psi(t)=t^n$, then $\Psi(k)=\delta(k-n)$, δ is the Kronecker delta symbol.

If $\psi(t)=e^{\lambda t}$, then $\Psi(k)=\frac{\lambda^k}{k!}$.

If $\psi(t)=g(t)\int_0^t h(s)ds$, then $\Psi(k)=\frac{G(k-1)}{k}$, where $k\geq 1$.

If $\psi(t)=\prod_{i=1}^{n} y_i(t)$, then

$$\Psi(k)=\sum_{r_1=0}^{k}\sum_{r_2=0}^{k-r_1}\cdots\sum_{r_n=0}^{k-r_1-\cdots-r_{n-1}} Y_1(r_1)\cdots Y_{n-1}(r_{n-1})Y_n(k-r_1-\cdots-r_n).$$

The proof of Theorem 2.1 is given in [19].

Theorem 2.2. Assume that $\Phi(k)$, $Y(k)$ and $Y_i(k)$ are the differential transformations of the functions $\phi(t)$, $y(t)$ and $y_i(t)$, respectively, and $q, q_i\in(0,1)$, $i=1,2$. Then:

(i) If $\phi(t)=y(qt)$, then $\Phi(k)=q^k Y(k)$.

(ii) If $\phi(t)=y_1(q_1 t)y_2(q_2 t)$, then $\Phi(k)=\sum_{l=0}^{k} q_1^l q_2^{k-l} Y_1(l)Y_2(k-l)$.

(iii) If $\phi(t)=\frac{d^m}{dt^m}y(qt)$, then $\Phi(k)=\frac{(k+m)!}{k!}q^{k+m}Y(k+m)$.

(iv) If $\phi(t)=\frac{d^n}{dt^n}y_1(q_1 t)\frac{d^m}{dt^m}y_2(q_2 t)$, then

$$\Phi(k)=\sum_{l=0}^{k} q_1^{l+n} q_2^{k-l+m}\frac{(l+n)!(k-l+m)!}{l!(k-l)!}Y_1(l+n)Y_2(k-l+m).$$

The proof of Theorem 2.2 is given in [20].

Theorem 2.3. Assume that $\Omega(k)$, $Y(k)$ and $Y_i(k)$ are the differential transformations of the function $\omega(t)$, $y(t)$ and $y_i(t)$, respectively, and $r, q, q_i\in(0,1)$, $i=1,2$. Then:

(I) If $\omega(t)=\int_0^{rt} y(qs)ds$, then $\Omega(k)=\frac{1}{k}r^k q^{k-1}Y(k-1)$

(II) If $\omega(t)=\int_0^{rt} y_1(q_1 s)y_2(q_2 s)ds$, then $\Omega(k)=\frac{1}{k}\sum_{l=0}^{k-1} r^k q_1^l q_2^{k-l-1}Y_1(l)Y_2(k-l-1)$

(III) If $\omega(t)=y(qt)\int_0^{rt} y_1(q_1 s)y_2(q_2 s)ds$, then

$$\Omega(k)=\sum_{l=0}^{k-1}\sum_{s=0}^{k-l-1}\frac{1}{k-l}r^{k-l}q^l q_1^s q_2^{k-l-s-1}Y(l)Y_1(s)Y_2(k-l-s-1),$$

where $k\in\mathbb{N}$. The proof of Theorem 2.3 is given in [20].

Description of the Method

In this section, an efficient and reliable algorithm is introduced to calculate the differential transform of the nonlinear function $\omega(u)$. This nonlinear function can be decomposed as

$$\omega(u)=\sum_{n=0}^{\infty} A_n \quad (5)$$

where A_n, $n\geq 0$ are the Adomian polynomials determined formally as follows [8, 21].

$$A_n=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left[\omega\left(\sum_{i=0}^{\infty}\lambda_i u_i\right)\right]\right]_{\lambda=0}, \quad (6)$$

The Adomian polynomials of $\omega(u)$ are introduced as

$A_0=\omega(u_0)$,

$A_1=u_1\omega^{(1)}(u_0)$,

$A_2 = u_2\omega^{(1)}(u_0) + \frac{1}{2!}u_1^2\omega^{(2)}(u_0),$

$A_3 = u_3\omega^{(1)}(u_0)) + u_1u_2\omega^{(2)}(u_0) + \frac{1}{3!}u_1^3\omega^{(3)}(u_0),$

(7) $A_4 = u_4\omega^{(1)}(u_0) + (u_1u_3 + \frac{1}{2!}u_2^2)\omega^{(2)}(u_0) + \frac{1}{2!}u_1^2u_2\omega^{(3)}(u_0) + \frac{1}{4!}u_1^4\omega^{(4)}(u);$

$$A_5 = u_5\omega^{(1)}(u_0) + (u_2u_3 + u_1u_4)\omega^{(2)}(u_0) + \frac{1}{2!}(u_1^2u_3 + u_1u_2^2)\omega^{(3)}(u_0) + \frac{1}{3!}u_1^3u_2\omega^{(4)}(u_0) + \frac{1}{5!}u_1^5\omega^{(5)}(u_0),$$

etc. Hence, the differential transform components of $\omega(u)$ are computed by utilizing their properties, they can be written in the following form (for $x = 0$)

$\Omega(0) = \omega(U(0)),$

$\Omega(1) = U(1)\omega^{(1)}(U(0)),$

$\Omega(2) = U(2)\omega^{(1)}(U(0)) + \frac{1}{2!}U^2(1)\omega^{(2)}(U(0)),$

$\Omega(3) = U(3)\omega^{(1)}(U(0)) + U(1)U(2)\omega^{(2)}(U(0)) + \frac{1}{3!}U^3(1)\omega^{(3)}(U(0)),$

$$\Omega(4) = U(4)\omega^{(1)}(U(0)) + (U(1)U(3) + \frac{1}{2!}U^2(2))\omega^{(2)}(U(0)) + \frac{1}{2!}U^2(1)U(2)\omega^{(3)}(U(0)) + \frac{1}{4!}U^4(1)\omega^{(4)}(U(0)),$$

$\Omega(5) = U(5)\omega^{(1)}(U(0)) + (U(2)U(3) + U(1)U(4))\omega^{(2)}(U(0)) + \frac{1}{2!}(U^2(1)U(3) + U(1)U^2(2))\omega^{(3)}(U(0) + \frac{1}{3!}U^3(1)U(2)\omega^{(4)}\omega(U(0)) + \frac{1}{5!}U^5(1)\omega^{(5)}(U(0)),$ (8)

and so on. The advantage of using this algorithm in comparison to the algorithm proposed in [22] for computation of differential transformation of a nonlinear function is that the algorithm directly deals with the nonlinear function of the problem in hand in its form with no differentiation, algebraic manipulation and no need to compute the differential transform other functions to obtain the required one.

Applications and Numerical Results

In this section, the proposed method is implemented on different examples with different nonlinearity types.

Example 4.1. Consider the following delay differential equation of the third order:

$y'''(t) = -1 + 2y^2(\frac{t}{2}), y(0) = 0, y'(0) = 1, y''(0) = 0.$
(9)

Substituting $t = 0$ into equation (9), we have $y'''(0) = -10$. Using the differential transformation method, the differential transform version of equation (9), we get

$(k+3)(k+2)(k+1)Y(k+3) = -\delta(k) + 2 \times \frac{1}{2^k}\Omega(k)$
(10)

where $\Omega(k)$ are the differential transform (Adomian polynomials) of the nonlinear $\omega(y) = y^2(t)$, and $Y(0) = 0$, $Y(1) = 1$, $Y(2) = 0$. Using the relation in (8), the Adomian polynomials for this nonlinear function are

$\Omega(0) = \omega(Y(0)) = 0, Y(3) = -\frac{1}{3!},$

$\Omega(1) = Y(1)\omega^{(1)}(Y(0)) = 0, Y(4) = 0,$

$\Omega(2) = Y(2)\omega^{(1)}(Y(0)) + \frac{1}{2!}Y^2(1)\omega^{(2)}(Y(0)) = 1,\ Y(5) = \frac{1}{5!},$

$\Omega(3) = Y(3)\omega^{(1)}(Y(0)) + Y(1)Y(2)\omega^{(2)}(Y(0)) + \frac{1}{3!}Y^3(1)\omega^{(3)}(Y(0)) = 0, Y(6) = 0,$

$\Omega(4) = Y(4)\omega^{(1)}(Y(0)) + \left(Y(1)Y(3) + \frac{1}{2!}Y^2(2)\right)\omega^{(2)}(Y(0)) + \frac{1}{2!}Y^2(1)Y(2)\omega^{(3)}(Y(0)) + \frac{1}{4!}Y^4(1)\omega^{(4)}(Y(0)) = -\frac{1}{3}, \quad Y(7) = -\frac{1}{7!}$ (11)

Using the recurrence relation (10) and the Adomian polynomials (11), Y(k) are evaluated. Hence, using inverse transformation in equation (4.2), the following series solution can be obtained

$$y(t) = t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \cdots + \frac{(-1)^n}{(2n+1)!}t^{2n+1} + \cdots\cdots$$

The closed form of the above series solution is $y(t) = \sin t$ which is the exact solution of equation (9).

Example 4.2. Consider the nonlinear pantograph-type integro-differential equation of the first order

$y'(t) + (\frac{1}{2}t - 2)y(t) - 2\int_0^t y^2(\frac{s}{2})ds = 1,$
(12)

with initial condition $y(0) = 0$. Substituting $t = 0$ in equation (12), we get $y'(0) = 1$. Applying the differential transformation method to equation (12), we get

$$(k+1)Y(k+1) + \sum_{l=0}^{k}(\tfrac{1}{2}\delta(l-1) - 2\delta(l))Y(k-1) - \frac{1}{2^{k-2}}\Omega(k-1) = \delta(k). \qquad (13)$$

where $\Omega(k)$ are the differential transform (Adomian polynomials) of the nonlinear $\omega(y) = y^2(t)$, and $Y(0) = 0$, $Y(1) = 1$.

Using the relations in (8) the Adomian polynomials for this nonlinear function are

$$\Omega(0) = \omega(Y(0)) = 0, Y(2) = 1,$$

$$\Omega(1) = Y(1)\omega^{(1)}(Y(0)) = 0, Y(3) = \frac{1}{2!},$$

$$\Omega(2) = Y(2)\omega^{(1)}(Y(0)) + \frac{1}{2!}Y^2(1)\omega^{(2)}(Y(0)) = 1,\ Y(4) = \frac{1}{3!},$$

$$\Omega(3) = Y(3)\omega^{(1)}(Y(0)) + Y(1)Y(2)\omega^{(2)}(Y(0)) + \frac{1}{3!}Y^3(1)\omega^{(3)}(Y(0)) = 2, Y(5) = \frac{1}{4!}, \qquad (14)$$

Using the recurrence relation (13) and the Adomian polynomials (14), $Y(k)$ are evaluated. Hence using inverse transformation in the equation, the following series solution can be obtained $y(t) = t + t^2 + \frac{1}{2!}t^3 + \frac{1}{3!}t^4 + \frac{1}{4!}t^5 + \cdots + \frac{1}{k!}t^{k+1} + \cdots$.

The closed form of the above series solution is $y(t) = te^t$, which is the exact solution of equation (12).

Example 4.3. Consider the following nonhomogeneous first-order integro-differential equation with proportional delay:

$$y'(t) - y(\tfrac{t}{2}) - \tfrac{1}{2}y(\tfrac{t}{2})\int_0^{\frac{t}{3}} y(s)y(\tfrac{s}{2})ds = 4 - 2t - \tfrac{8}{81}t^4, \qquad (15)$$

$y(0) = 0$. Substituting $t = 0$ in equation (15), we obtain the second conditions $y'(0) = 4$. Applying the differential transformation method to equation (15), we get

$$(y+1)Y(k+1) - \frac{1}{2^k}Y(k) - \frac{1}{2}\sum_{l=0}^{k-1}\sum_{s=0}^{k-l-1}\frac{1}{k-l}(\frac{1}{3})^{k-l}(\frac{1}{2})^{l}Y(l)\Omega(s) = 4\delta(k) - 2\delta(k-1) - \frac{8}{81}\delta(k-4) \qquad (16)$$

where $\Omega(s)$ are the differential transform (Adomian polynomials) of the nonlinear $\omega(y) = y(t)y(\frac{t}{2})$, and $Y(0) = 0$, $Y(1) = 4$. Using the relations in (8), the Adomian polynomials for this nonlinear function are

$$\Omega(0) = \omega(Y(0)) = 0,$$

$$\Omega(1) = Y(1)\omega^{(1)}(Y(0)) = 0, Y(2) = 0,$$

$$\Omega(2) = Y(2)\omega^{(1)}(Y(0)) + \frac{1}{2!}Y^2(1)\omega^{(2)}(Y(0)) = 128, Y(3) = 0,$$

$$\Omega(3) = Y(3)\omega^{(1)}(Y(0)) + Y(1)Y(2)\omega^{(2)}(Y(0)) + \frac{1}{3!}Y(1)\omega^{(3)}(Y(0)) = 0, Y(4) = 0, \qquad (17)$$

Similarly, we obtain $Y(k) = 0,\ k \geq 2,\ \Omega(m) = 0,\ m \geq 3$. Using the Adomian polynomials (17) and the recurrence relation (16), $Y(k)$ is assessed. So, the series solution of $y(t) = 4t$ can be obtained using inverse transformation in equation (4.2).

Conclusion

In this study, we showed that the differential transformation method can be utilized successfully for solving nonlinear integro-differential and differential equations with proportional delays. In this paper, we presented a new approach for applying the differential transform method for solving nonlinear differential and integro-differential equations with proportional delays. In the recurrence relation, the differential transform of the nonlinear term is replaced by its Adomian polynomial of index k. Therefore, the dependent variable components are replaced by their corresponding differential transforms of the same index k. The major advantage of this method is that it can be directly applied to functional integro-differential and differential equations with no require to perturbation, discretization, or linearization. In addition, this method is able to remarkably reduce the size of computational work.

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